Appendix E

Parametrized Curves and Arclength

For many purposes, the precise parametrization of a curve σ is not important, in the sense that some property of the curve that we are interested in is unchanged if we "reparametrize" the curve. Let us look at just what reparametrization means. Suppose that t is a C^1 function with a strictly positive derivative on a closed interval $[\alpha, \beta]$. Then t is strictly monotonic, and hence it maps $[\alpha, \beta]$ one-to-one onto some other closed interval $[a, b]$. Thus if $\sigma : [a, b] \to \mathbf{R}^n$ is a C^1 parametrized curve, then $\tilde{\sigma} = \sigma \circ t : [\alpha, \beta] \to \mathbf{R}^n$ is another C^1 parametrized curve which clearly has the same image as σ and is called the *reparametrization* of σ defined by the parameter transformation t. (If you like, you can think of t as a "variable that parameterizes the points of the interval $[\alpha, \beta]$ by points of the interval $[a, b]$" and with this interpretation σ and $\tilde{\sigma}$ become "the same".) In particular, given any interval $[\alpha, \beta]$, we always find an affine map $t(\tau) = c\tau + k$ that maps it onto $[a, b]$, so reparametrization allows us to adjust a parameter interval as convenient in situations where parametrization is not relevant.

A reparametrization of $\sigma : [a, b] \to \mathbf{R}$ can always be thought of as arising by starting from a positive, continuous function $\rho : [a, b] \to \mathbf{R}$ and letting t be the inverse function of its indefinite integral, τ. In fact $\tau(t) := \int_a^t \rho(\xi)\, d\xi$ is a smooth C^1 function with a positive derivative, so it does indeed map $[a, b]$ one-to-one onto some interval $[\alpha, \beta]$, and by the inverse function theorem $t := \tau^{-1} : [\alpha, \beta] \to \mathbf{R}$ is C^1 with a positive derivative. A very important special case of this is reparametrization by arclength. Suppose that σ is nonsingular, i.e.,

σ' never vanishes. Define $s : [a, b] \to \mathbf{R}$ by $s(t) := \int_a^t \|\sigma'(\xi)\| \, d\xi$, and recall that by definition it gives the arclength along σ from a to t. This is a smooth map with positive derivative $\|\sigma'(t)\|$ mapping $[a, b]$ onto $[0, L]$, where L is the length of σ. The inverse function, $t(s)$, mapping $[0, L]$ to $[a, b]$, gives the point of $[a, b]$ where the arclength of σ measured from its left endpoint is s, and the curve $s \mapsto \sigma(t(s))$ is a reparametrization of σ called its reparametrization by arclength. More generally, we say that a curve $\sigma : [a, b] \to \mathbf{R}^n$ is *parameterized by arclength* if the length of σ between $\sigma(a)$ and $\sigma(t)$ is equal to $t - a$, and we say that σ is *parametrized proportionally to arclength* if that length is proportional to $t - a$.

▷ **Exercise E–1.** Show that the length of a curve is unchanged by reparametrization. (Hint: This follows from a combination of the chain rule and the change of variables formula for an integral.)

▷ **Exercise E–2.** Show that a curve σ is parametrized proportionally to arclength if and only if $\|\sigma'(t)\|$ is a constant, and it is parametrized by arclength if and only if that constant equals one.

▷ **Exercise E–3.** Prove the old saying, "A straight line is the shortest distance between two points." That is, if $\sigma : [a, b] \to \mathbf{R}^n$ is a C^1 path of length L, then $\|\sigma(b) - \sigma(a)\| \leq L$, with equality if and only if σ is a straight line from $\sigma(a)$ to $\sigma(b)$. (Hint: As we have just seen, we can assume without loss of generality that σ is parametrized proportionally to arclength, i.e., that $\|\sigma'(t)\|$ is a constant. Let $v := \sigma(b) - \sigma(a)$, so that what we must show is that $\|v\| \leq L$ with equality if and only if σ' is a constant. If $v = 0$, i.e., if $\sigma(b) = \sigma(a)$, the result is trivial so we can assume $v \neq 0$ and define a unit vector $e = \frac{v}{\|v\|}$, so that $\|v\| = \langle v, e \rangle$. Now $v = \int_a^b \sigma'(t) \, dt$, and since e is a constant vector, $\|v\| = \langle v, e \rangle = \int_a^b \langle \sigma'(t), e \rangle \, dt$. Finally note that by the Schwarz Inequality, $\langle \sigma'(t), e \rangle \leq \|\sigma'(t)\|$ and equality holds for all t if and only if $\sigma'(t)$ is a multiple of e for each t, and this multiple must be a constant since $\|\sigma'(t)\|$ is a constant.)