Appendix J

Iterative Interpolation and Its Error

In this appendix we give a brief review of iterative polynomial interpolation and corresponding error estimates used in the development and analysis of numerical methods for differential equations.

The unique polynomial of degree n,

$$p_{x_0,\ldots,x_n}(x) = \sum_{j=0}^{n} a_j x^j, \tag{J.1}$$

that interpolates a function $f(x)$ at $n+1$ points,

$$p_{x_0,\ldots,x_n}(x_i) = y_i = f(x_i), \quad 0 \le i \le n, \tag{J.2}$$

can be found by solving simultaneously the $(n+1) \times (n+1)$ linear system of equations for the $n+1$ unknown coefficients a_j given by (J.2). It can also be found using Lagrange polynomials

$$p_{x_0,\ldots,x_n}(x) = \sum_{i=0}^{n} y_i L_{i,x_0,\ldots,x_n}(x) \tag{J.3}$$

where

$$L_{i,x_0,\ldots,x_n}(x) = \prod_{0 \le j \le n, j \ne i} \frac{(x - x_j)}{(x_i - x_j)}. \tag{J.4}$$

Here, we develop $p_{x_0,\ldots,x_n}(x)$ inductively, starting from $p_{x_0}(x) = y_0$ and letting

$$\begin{aligned} &p_{x_0,\ldots,x_{j+1}}(x) \\ &\quad = p_{x_0,\ldots,x_j}(x) + c_{j+1}(x - x_0)\cdots(x - x_j), \quad j = 0,\ldots,n-1 \end{aligned} \tag{J.5}$$

(so that each successive term does not disturb the correctness of the prior interpolation) and defining c_{j+1} so that $p_{x_0,\dots,x_{j+1}}(x_{j+1}) = y_{j+1}$, i.e.,

$$c_{j+1} = \frac{y_{j+1} - p_{x_0,\dots,x_j}(x_{j+1})}{(x_{j+1} - x_0)} = f[x_0, \dots, x_{j+1}]. \tag{J.6}$$

Comparing (J.6) with (J.3), (J.4) gives an alternate explicit expression for $f[x_0, \dots, x_n]$, the leading coefficient of the polynomial of degree n that interpolates f at $x_0, \dots, x_n$:

$$f[x_0, \dots, x_n] = \sum_{i=0}^{n} \frac{f(x_i)}{\prod_{j \neq i}(x_i - x_j)} \tag{J.7}$$

from which follows the divided difference relation

$$f[x_0, \dots, x_n] = \frac{f[x_0, \dots, \hat{x_j}, \dots x_n] - f[x_0, \dots, \hat{x_i} \dots, x_n]}{x_i - x_j} \tag{J.8}$$

(where ˆ indicates omission).

For our purposes, we want to estimate $p_{x_0,\dots,x_n}(t) - f(t)$, and to do so, we simply treat t as the next point at which we wish to interpolate f in (J.5):

$$p_{x_0,\dots,x_n}(t) + f[x_0, \dots, x_n, t](t - x_0) \cdots (t - x_n) = f(t)$$

or

$$p_{x_0,\dots,x_n}(t) - f(t) = f[x_0, \dots, x_n, t](t - x_0) \cdots (t - x_n). \tag{J.9}$$

Finally, we estimate the coefficient $f[x_0, \dots, x_n, t]$ using several applications of Rolle's Theorem. Since $p_{x_0,\dots,x_n,t}(x) = f(x)$ or $p_{x_0,\dots,x_n,t}(x) - f(x) = 0$ at $n + 2$ points $x_0, \dots, x_n, t$, Rolle's Theorem says that $p'_{x_0,\dots,x_n,t}(x) - f'(x) = 0$ at $n + 1$ points, one in each open interval between consecutive distinct points of $x_0, \dots, x_n, t$. Repeating this argument, $p''_{x_0,\dots,x_n,t}(x) - f''(x) = 0$ at n points on the intervals between the points described in the previous stage, and repeating this $n - 1$ more times, there is one point ξ in the interior of the minimal closed interval containing all of the original points $x_0, \dots, x_n, t$ at which

$$p^{(n+1)}_{x_0,\dots,x_n,t}(\xi) - f^{(n+1)}(\xi) = 0. \tag{J.10}$$

But because $f[x_0, \ldots, x_n, t]$ is the leading coefficient of the polynomial $p_{x_0,\ldots,x_n,t}(x)$ of degree $n+1$ that interpolates f at the $n+2$ points $x_0, \ldots, x_n, t$, if we take $n+1$ derivatives, we are left with a constant, that leading coefficient times $(n+1)!$:

$$p^{(n+1)}_{x_0,\ldots,x_n,t}(x) = (n+1)! f[x_0, \ldots, x_n, t]. \tag{J.11}$$

Combining this with (J.10) gives

$$f[x_0, \ldots, x_n, t] = \frac{f^{(n+1)}(\xi)}{(n+1)!} \tag{J.12}$$

where ξ in the interior of the minimal closed interval containing all of the original points $x_0, \ldots, x_n, t$, and substituting into (J.9) yields the basic interpolation error estimate:

$$p_{x_0,\ldots,x_n}(t) - f(t) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(t - x_0)\cdots(t - x_n). \tag{J.13}$$

For $n = 0$ this recovers the mean value theorem

$$\frac{f(t) - f(x_0)}{t - x_0} = f'(\xi) \tag{J.14}$$

for some $\xi \in (x_0, t)$.

Since many multistep methods involve simultaneous interpolation of y and y' at $t_n, \ldots, t_{n-m+1}$, to treat these, we would want to have the corresponding estimates for osculatory interpolation that can be obtained by letting pairs of interpolation points coalesce. In the simplest cases, for two points, this process recovers the tangent line approximation and estimate

$$f[x_0] + f[x_0, x_0](x - x_0) = f(x_0) + f'(x_0)(x - x_0).$$

For four points, it recovers the cubic spline interpolation approximating a function and its derivative at two points.