Appendix F

Smoothness with Respect to Initial Conditions

Suppose that V is a C^1 vector field on $\mathbf{R}^n$ and assume that the maximal solution σ_p of $\frac{dx}{dt} = V(x)$ is defined on $I = [a, b]$. For each $x \in \mathbf{R}^n$, the differential of V at x is a linear map $DV_x : \mathbf{R}^n \to \mathbf{R}^n$, and it is continuous in x since V is C^1. Thus $A(t) = DV_{\sigma_p(t)}$ defines a continuous map $A : I \to \mathbf{L}(\mathbf{R}^n)$. The differential equation $\frac{dx}{dt} = A(t)x$ is an example of the nonautonomous linear ODEs studied in Section 2.2. It is called the *variational equation* associated to the solution σ. By the general theory of such equations developed in Chapter 2, we know that for each ξ in $\mathbf{R}^n$, the variational equation will have a unique solution $u(t, \xi)$ defined for $t \in I$ and satisfying the initial condition $u(t_0, \xi) = \xi$. For each t in I, the map $\xi \mapsto u(t, \xi)$ is a linear map of $\mathbf{R}^n$ to itself that we will denote by $\delta\sigma_p(t)$. What we are going to see next is that the map $(t, p) \mapsto \sigma_p(t)$ is C^1 and that $\delta\sigma_p(t)$ is the differential at p of the map $q \mapsto \sigma_q(t)$ of $\mathbf{R}^n$ to itself. (Note that the derivative of $\sigma_p(t)$ with respect to t obviously exists and is continuous since $\sigma_p(t)$ satisfies $\sigma_p'(t) = V(\sigma_p(t))$.)

$\triangleright$ **Exercise F–1.** Check that *if* $q \mapsto \sigma_q(t)$ is indeed differentiable at p, then its differential must in fact be $\delta\sigma_p(t)$. Hint: Calculate the differential of both sides of the differential equation with respect to p to see that $D\sigma_p(t)(\xi)$ satisfies the variational equation. On the right side of the equation use the chain rule and on the left side interchange the order of differentiation.

Recall that (by definition of the differential of a mapping) in order to prove that $q \mapsto \sigma_q(t)$ is differentiable at p and that $u(t,\xi) = \delta\sigma_p(t)(\xi)$ is its differential at p in the direction ξ, what we need to show is that if $g(t) := \|(\sigma_{p+\xi}(t) - \sigma_p(t)) - u(t,\xi)\|$, then $\frac{1}{\|\xi\|}g(t)$ goes to zero with $\|\xi\|$. What we will show is that there are fixed positive constants C and M such that for any positive ϵ there exists a δ so that $g(t) < C\epsilon\,\|\xi\|\,e^{Mt}$ provided $\|\xi\| < \delta$, which clearly implies that $\frac{1}{\|\xi\|}g(t)$ goes to zero with $\|\xi\|$, uniformly in t. To prove the latter estimate, it will suffice by Gronwall's Inequality to show that $g(t) < C\epsilon\,\|\xi\| + M\int_0^t g(s)\,ds$.

▷ **Exercise F–2.** Derive this integral estimate. Hint: $\sigma_{p+\xi}(t) = p+\xi+\int_0^t V(\sigma_{p+\xi}(s))\,ds$ and $\sigma_p(t) = p+\int_0^t V(\sigma_p(s))\,ds$, while $u(t,\xi) = \xi + \int_0^t DV_{\sigma_p(s)} u(s,\xi)\,ds$. Taylor's Theorem with Remainder gives $V(q+x) - V(q) = DV_q(x) + \|x\|\,r(q,x)$ where $\|r(q,x)\|$ goes to zero with x, uniformly for q in some compact set. Take $q = \sigma_p(s)$ and $x = \sigma_{p+\xi}(s) - \sigma_p(s)$ and verify that

$$g(t) = \|\xi\| \int_0^t \rho(\sigma_p(s), \sigma_{p+\xi}(s) - \sigma_p(s))\,ds + \int_0^t DV_{\sigma_p(s)} g(s)\,ds.$$

Now choose $M = \sup_{s\in I}\left\|DV_{\sigma_p(s)}\right\|$ and recall that from the theorem on continuity with respect to initial conditions we know that $\|\sigma_{p+\xi}(s) - \sigma_p(s)\| < \|\xi\|\,e^{Ks}$. The rest is easy, and we have now proved the case $r = 1$ of the following theorem.

F.0.1. Theorem on Smoothness w.r.t. Initial Conditions. *Let V be a C^r vector field on $\mathbf{R}^n$, $r \geq 1$, and let $\sigma_p(t)$ denote the maximal solution curve of $\frac{dx}{dt} = V(x)$ with initial condition p. Then the map $(p,t) \mapsto \sigma_p(t)$ is C^r.*

▷ **Exercise F–3.** Prove the general case by induction on r. Hint: As we saw, the first-order partial derivatives are solutions of an ODE whose right-hand side is of class C^{r-1}.